Groebner Bases, Toric Ideals and Integer Programming: An Application to Economics

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History

- Groebner bases were developed by Buchberger in 1965, who later named them after his advisor, Wolfgang Groebner.
- Groebner bases were first used to solve the Ideal Membership Problem.
- Groebner bases can be described as special generating subsets of an ideal in a polynomial ring.
- They provide the foundation for many algorithms in algebraic geometry and commutative algebra.

Efficient Generation of Polynomial Ideals and an Application to Integer Programming

- Polynomial rings $k\left[x_1, \ldots\ldots, x_n\right]$ (in a finite number of indeterminates over a field) are the most fundamental objects of Computational Commutative Algebra.
- Polynomial rings are associative, **commutative** rings with **identity**.

Ideals

- A subset of a ring, $I \subset k[x_1, \ldots\ldots, x_n]$, is an ideal if it satisfies:
   i. $0 \in f$.
   ii. If $f, g \in I$, then $f + g \in I$.
   iii. If $f \in I$ and $h \in k[x_1, \ldots., x_n]$, then $hf \in I$.

Different examples of ideals in the polynomial ring:

- Ideal generated by a finite number of polynomials.
- Ideal generated by monomials (LT ideal)
- Ideal generated by binomials (toric ideal).
- Ideal generated by varieties.

Ideal Membership Problem

In the single variable case:

- Given $f \in k[x]$, and an ideal I=$< f_1,\ldots\ldots, f_s >$, how to determine whether a given polynomial f in $k[x]$ lies, for example, in $< x^4 - 1, x^6 - 1 >$?

  - How to find the generator of the ideal contained in *k*[x]?
  - GCD is the key to the ideal membership problem. In the single variable case, if the division of the polynomial yields zero as the remainder, then it is in the given ideal.

Polynomials of one variable

- Given a nonzero polynomial $f \in k\left[x\right]$, let

$$f = a_0 x^m + a_1 x^{m-1} + ... + a_m$$, where $$LT\left(f\right) = a_0 x^m$$

- Division algorithm- let k be a field and let g be a nonzero polynomial in k[x]. Then every f in k[x] can be written as

f= qg + r, where q, r in K[x], and are unique.

0< deg(r) < deg (g)

Ex: $$x^3 + 2x^2 + x + 1 = \left(\tfrac{1}{2} x^2 + \tfrac{3}{4} x\right)\left(2x+1\right) + \left(\tfrac{1}{4} x + 1\right)$$

Greatest Common Divisor and The Euclidean Algorithm

- The greatest common divisor of polynomials $f_1, \ldots., f_s \in k[x]$ is a polynomial h such that:
  1. h divides $f_1, \ldots., f_s$
  2. if p is another polynomial which divides $f_1, \ldots., f_s$, then p divides h.

- GCD gives the generator for the ideal $< f_1, \ldots., f_s >$. It exists and is unique, as the consequence of the division algorithm.
- The Euclidean Algorithm is the classic algorithm for computing the GCD of two polynomials in k[x]. It can also be modified to calculate the GCD of multivariable polynomials.

Multivariable Polynomials

- A polynomial f in K[$x_1,\ldots,x_n$] with coefficients in K is a finite linear combination of monomials.

$$f = \sum_{\alpha} a_\alpha x^\alpha, a^\alpha \in k$$

  - $multi\deg(f) = \max\left(\alpha \in \mathbb{Z}^n_{\geq 0}\right)$
  - $LC(f) = a_{multi\deg(f)} \in k$
  - $LM(f) = x^{multi\deg(f)}$
  - $LT(f) = LC(f) \bullet LM(f)$

Monomial

- A monomial in K$[x_1,....,x_n]$ is a product of the form $x_1^{\alpha_1} x_2^{\alpha_2} .....x_n^{\alpha_n}$ denoted x^a and $a = (\alpha_1,....,\alpha_n)$ is the exponent vector with

  $a_i \in Z_{\geq 0}$

- The total degree of this monomial is the sum

$$\alpha_1 + ... + \alpha_n$$

Term Ordering

- A monomial (term) order on $k[x_1,....,x_n]$ is a relation $\prec$ on a set of monomials x^a or, equivalently, on the set of exponent vectors **a** $a=(\alpha_1,....,\alpha_n)$, $a_i \in Z_{\geq 0}$ such that:
  a) $\prec$ is a total order;
  b) If $x^u \prec x^v \Rightarrow x^{u+w} \prec x^{v+w}$ $w \in \mathbb{Z}^n_{\geq 0}$
  c) $\prec$ is a well ordering.
- Given a term order $\prec$, every nonzero polynomial $f \in k[x_1,....,x_n]$ has a unique leading monomial denoted $in_{\prec}(f)$.

Example

How to order $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2 \in k[x, y, z]$?

1. With respect to the lex order, the reordering of the terms of f in decreasing order is:
$$f = -5x^3 + 7x^2z^2 + 4xy^2z + 4z^2$$
2. With respect to grlex order:
$$f = 7x^2z^2 + 4xy^2z - 5x^3 + 4z^2$$
3. With respect to grevlex order:
$$f = 4xy^2z + 7x^2z^2 - 5x^3 + 4z^2$$

Lexicographic Order

- Let $\alpha = (\alpha_1, \ldots., \alpha_n)$ and $\beta = (\beta_1, \ldots..\beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ the left most nonzero entry is positive. We will write $x^\alpha > x^\beta$ if $\alpha >_{lex} \beta$.

Ex: $(3,2,4) >_{lex} (3,2,1)$ since $\alpha - \beta = (0,0,3)$

$$x_1^3 x_2^2 x_3^4 >_{lex} x_1^3 x_2^2 x_3^1$$

Division algorithm in multivariable polynomials

- Fix a monomial order $\prec$ on $\mathbb{Z}_{\geq 0}^{n}$ and let $F=(f_1,....,f_s)$ be an ordered s-tuple of polynomials in $K[x_1,...,x_n]$. Then every f in $\mathrm{K}[x_1,...,x_n]$ can be written as

$$F = a_1 f_1 + ... + a_s f_s + r$$

Where:

1. The $a_i, r \in \mathrm{K}[x_1,...,x_n]$
2. No term of r is divisible by any $in_{\prec}(f_i)$.
3. $in_{\prec}(f) \geq in(a_i f_i)$ 1< i < s.

- Changing the order of polynomials in the division set can change the result and, in particular, the remainders will be different.

EX:

Division 1: $xy^2 - x \in \langle xy+1, y^2-1 \rangle$

$$f_1 = xy+1,\ f_2 = y^2-1$$

$$xy^2 - x = y\cdot(xy+1) + 0\cdot(y^2-1) - (x+y)$$

$$r = -x - y$$

Division 2: $f_1 = y^2-1,\ f_2 = xy+1$

$$xy^2 - x = x\cdot(y^2-1) + 0\cdot(xy+1) + 0$$

$$r = 0$$

- Ideal Membership problem: Given $f \in \mathrm{K}[x_1, \ldots, x_n]$ and an ideal $I = < f_1, \ldots., f_s >$, determine if $f \in I$

- It is impossible to use the method of the one variable case to solve the ideal membership problem.

Groebner Bases

Definition: Given a monomial order $\prec$ and an ideal $I \subset F[x_1,\ldots.,x_n]$. We say that $\{g_1,\ldots.,g_t\}$ is a Groebner Basis of I if

$$<in_{\prec}(g_1),\ldots,in_{\prec}(g_t)>=<in_{\prec}(I)> .$$

- Any Groebner basis for an ideal I is a generating set for I.

Groebner Bases

- If $\{g_1,....,g_t\}$ is a Groebner basis for I, and $f \in K[x_1,...,x_n]$, then f can be written **uniquely** in the form f= g+r, where $g \in I(g = a_1 g_1 + ... + a_t g_t)$ and no term of r is divisible by $in_{\prec}(g_i)$.
- If $\{g_1,....,g_t\}$ is a Groebner basis for I, and $f \in K[x_1,...,x_n]$, then $f \in I$ $\Leftrightarrow$ the remainder of f on division by $g_1,....,g_n$ is zero.

Hilbert Basis Theorem

- Every ideal $I \subset k[x_1,\ldots.,x_n]$ has a finite generating set. That is, $I = < g_1,\ldots.,g_t >$ for some $g_1,\ldots.,g_t \in I$
- Fix a monomial order. Then every ideal $I \subseteq K[x_1,\ldots.,x_n]$ other than {0} has a finite Groebner basis.

Buchberger Algorithm

Let $f, g \in \mathrm{K}[x_1, \ldots, x_n]$ be nonzero. Fix a monomial order and let $in_{\prec}(f) = cx^u, in_{\prec}(g) = dx^v$, $c, d \in k$. Let x^w be the least common multiple of x^u and x^v .

The S polynomial of f and g is:

$$S(f,g) = \frac{x^\gamma}{LT(f)} \cdot f - \frac{x^\gamma}{LT(g)} \cdot g$$

Example

$$f = x^3 - 2xy$$

$$g = x^2y - 2y^2 - x$$

$$S(f,g) = \frac{x^3y}{x^3}\cdot(x^3 - 2xy) - \frac{x^3y}{x^2y}\cdot(x^2y - 2y^2 - x)$$

$$= x^3y - \frac{2x^4y^2}{x^3} - x^3y + \frac{2x^3y^3}{x^2y} + \frac{x^4y}{x^2y}$$

$$= -2xy^2 + 2xy^2 + x^2 = x^2$$

Which is not divisible by $in_{\prec}(f)$ or by $in_{\prec}(g)$.

Buchberger Algorithm

Input: $F = (f_1, ..., f_s)$

Output: a Groebner basis $G = (g_1, ..., g_s)$ for I, with $F \subset G$

$G' := F$

REPEAT

$G' := G$

FOR each pair $\{p, q\}, p \neq q$ in G' DO

$S := \overline{S(p,q)}^{G'}$

IF $S \neq 0$ THEN $G := G \cup \{S\}$

UNTIL

$G = G'$

$r = \overline{f}^{G}$ is the unique remainder of f on division by $G = \{g_1, ..., g_n\}$ (r is **normal** form)

Example

$$G_1 = \{x^3 - 2xy, x^2y - 2y^2 - x, x^2\}$$

$$\overline{S(g_1, g_2)}^{G_1} = 0$$

$$\overline{S(g_1, g_3)}^{G_1} = -2xy = g_4$$

$$\overline{S(g_2, g_3)}^{G_1} = -x - 2y^2 = g_5$$

$$G_2 = \{x^3 - 2xy, x^2 - 2y^2 - x, x^2, -2xy, -x - 2y^2\}$$

REPEAT

$$\overline{S(g_1, g_5)}^{G_2} = -4y^3$$

$$\overline{S(g_4, g_5)}^{G_2} = -2y^3$$

$\overline{S(g_i, g_j)}^{G_2} = 0$ all other i,j

$$G_3 = \{x^3 - 2xy, x^2 - 2y^2 - x, x^2, -2xy, -x - 2y^2, y^3\}$$

REPEAT

$\overline{S(g_i, g_j)}^{G_3} = 0 \quad \forall i \neq j$

$$G_3 = G$$

My Research

- My research consists of applying specific concepts and constructions from **Commutative Algebra** to the area known in Economics as **Integer Programming**.
- In particular, by means of several algorithms (Conti-Traverso, Buchberger), a specific **Transportation Problem** (one of the classical problems of Integer Programming) is solved.
- To find the **Toric Ideal** and **Groebner Basis** needed to obtain the optimal solution, it was necessary to explore several versions of **Computational Commutative Algebra** software such as CoCoa and Maucalay.

Terminologies

- A ***homomorphism*** from a ring $(R, +, \cdot)$ to a ring $(S, ‡, *)$ is a function f from R to S that that preserves the structure of a ring, that is, for all a, b in R, the following identities hold:

- $f(a + b) = f(a)$ ‡ $f(b)$
- $f(a \cdot b) = f(a) * f(b)$

Toric Ideal

Fix $A = \{a_1, ..., a_n\} \subset Z^d$

Identify each a_i with a monomial

$t^{a_i} \in F[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}]$ (Laurent polynomial ring)

Consider the semigroup homomorphism:

$$\pi : N^n \to Z^d$$

$$u = (u_1, ..., u_n) \to u_1 a_1 + ... + u_n a_n$$

Which lifts to:

$$\hat{\pi} : k[x_1, ..., x_n] \to k[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}]$$

ker($\hat{\pi}$) is I_A , the Toric Ideal of A

Toric Ideal

- Toric ideal is a special class of ideals in the polynomial ring.
- The toric ideal I_A is a spanned as a F vector space by the set of binomials

$$\{x^u - x^v : u, v \in N^n and \pi(u) = \pi(v)\}$$

Integer Programming

- Given any "cost vector" $w \in R^n$, find a point **u** in $\pi^{-1}(b)$ which minimizes the value of the linear functional $u \rightarrow u \bullet w$.
- The idea here is to apply toric ideals to model and solve the problem.

Given any cost vector $w \in R^n$ find a point **u** in $\pi^{-1}(b)$ which minimizes the value of the linear functional.

$$u \mapsto u \bullet w$$

Fix $s, t \in Z^+$,

$$e_1, ..., e_s \in N^s$$

$$e_1^{'}, ..., e_t^{'} \in N^t, \quad n = s \bullet t, \quad d = s + t$$

$$A = \{e_i \oplus e_j^{'} : i = 1, ..., s,\ j = 1, ..., t\} \subset N^d$$

$$\pi : N^{s \times t} \to N^{s+t},$$

$$\{u_{ij}\} \mapsto (\sum_{j=1}^{t} u_{1j}, ..., \sum_{j=1}^{t} u_{sj}; \sum_{i=1}^{s} u_{i1}, ..., \sum_{i=1}^{s} u_{it})$$

The fiber $\pi^{-1}(r;c)$ consists of all non-negative integer $s \times t$ matrices with row sums **r** and column sums **c**.

Algorithm

Input: a $d \times n$ matrix A and a cost function $w \in R^n$

Output: An optimal point $u \in \pi^{-1}(b)$ with $u \bullet w$ minimal for any given $b \in im(\pi)$.

1. Compute the reduced Groebner basis G for with respect to weight order.
2. For any given vector $b \in im(\pi)$ do:

    a. Find any feasible solution $v \in \pi^{-1}(b)$.

    b. Compute the normal form x^u of x^v with respect to the weight order. Output **u.**

The Transportation Problem

s= 4 factories F_1, F_2, F_3, F_4

t=3 stores S_1, S_2, S_3

F_1 produces 120 units

F_2 produces 204 units

F_3 produces 92 units

F_4 produces 55 units

S_1 demands 183 units

S_2 demands 190 units

S_3 demands 98 units

W is the non negative real cost associated with transporting one unit from F_i to S_j.

We want the transportation plan that minimizes total cost of shipping all 942 units. In the fiber:

$$\pi^{-1}(120, 204, 92, 55; 183, 190, 98)$$

We have all the possible transportation plans.

$$w = \begin{matrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{matrix}$$

And let $\prec$ be the diagonal term order $(x_{11} < x_{12} < .... < x_{st})$.

In this case $G_{\prec w}$ is the set of 2*2 minors.

Consider the feasible solution

$$x^{v} = x_{11}^{120} x_{12}^{0} x_{13}^{0} x_{21}^{14} x_{22}^{190} x_{23}^{0} x_{31}^{0} x_{32}^{0} x_{33}^{92} x_{41}^{49} x_{42}^{0} x_{43}^{6}$$

Corresponding to the matrix v =

$$\begin{matrix} 120 & 0 & 0 \\ 14 & 190 & 0 \\ 0 & 0 & 92 \\ 49 & 0 & 6 \end{matrix}$$

The normal form of x^v with respect to the Groebner basis

$$\{x_{il}x_{jk} - x_{ik}x_{jl} : 1 \le i < j \le 4, 1 \le k < l \le 4\}$$ is

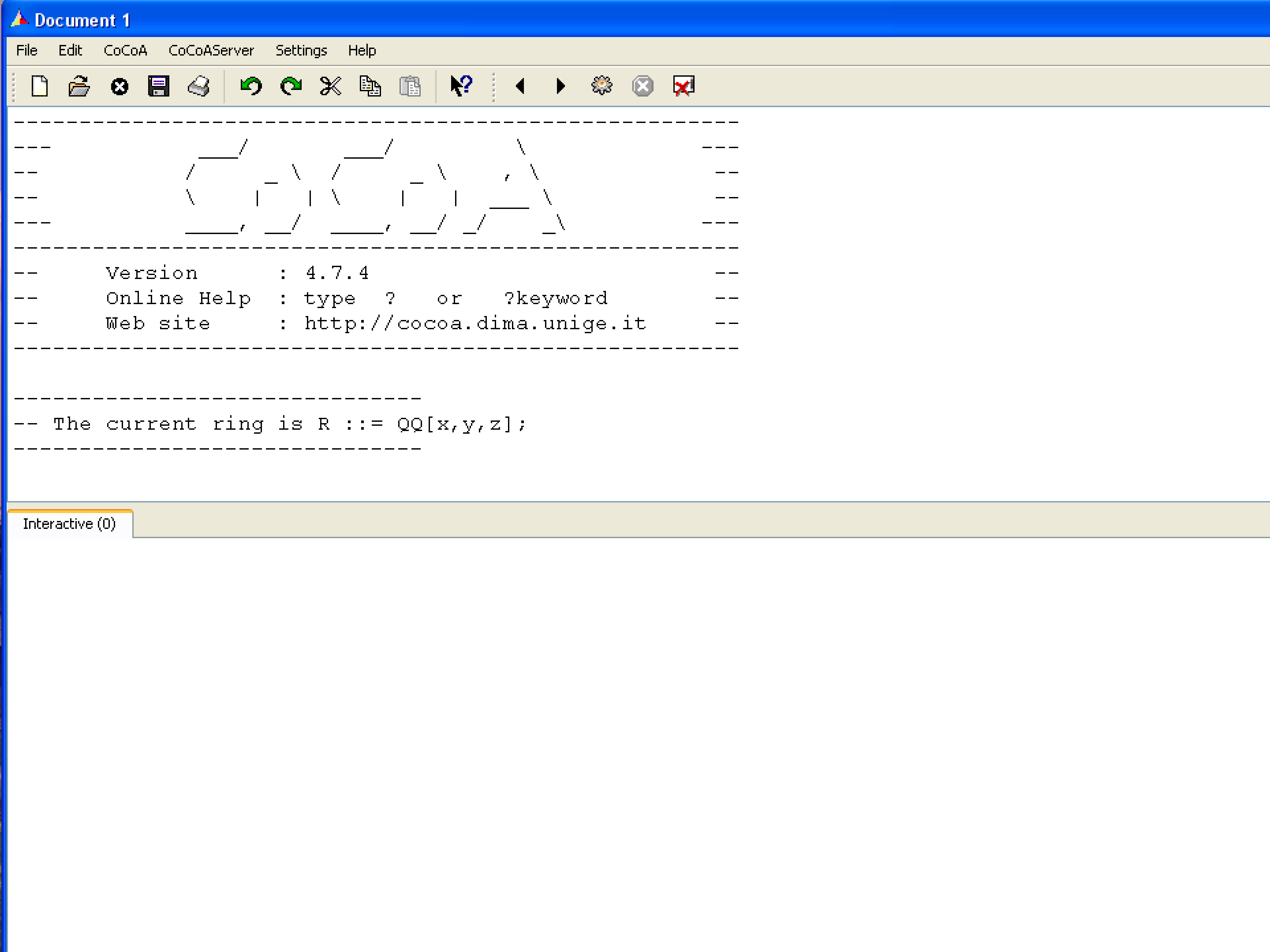
$$x^u = x_{11}^{120} x_{22}^{161} x_{23}^{43} x_{31}^{63} x_{32}^{29} x_{43}^{55}$$

Which is the optimal solution

$$u = \begin{matrix} 120 & 0 & 0 \\ 0 & 161 & 43 \\ 63 & 29 & 0 \\ 0 & 0 & 55 \end{matrix}$$

Use of Software

- CoCoa
- MaCaulay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-- Web site : http://cocoa.dima.unige.it --
-- The current ring is R ::= QQ[x,y,z];
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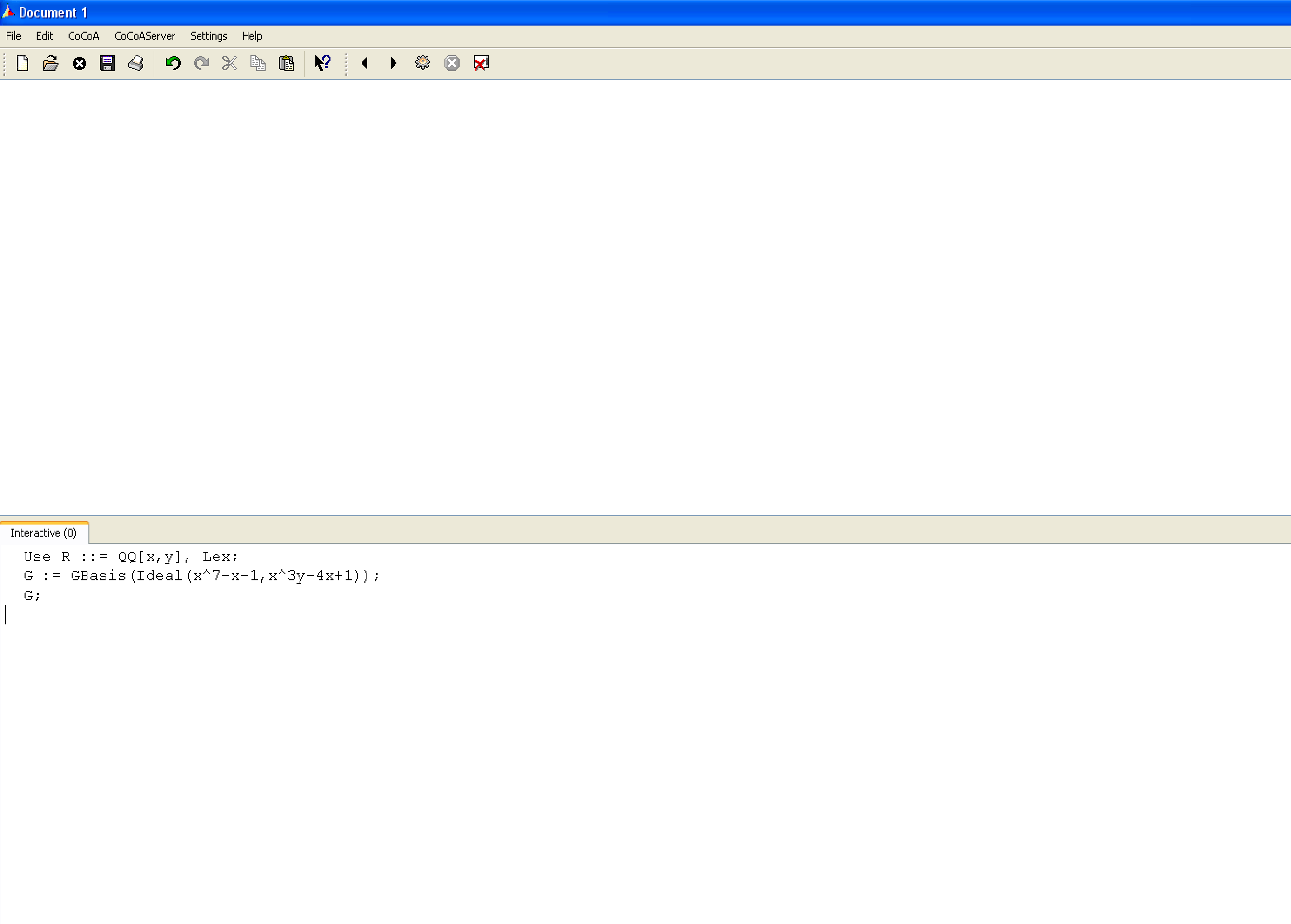
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Use R ::= QQ[x,y], Lex;
G := GBasis(Ideal(x^7-x-1,x^3y-4x+1));
G;

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```
Use R ::= QQ[x,y], Lex;
  Set Indentation;  -- to improve the appearance of the output
  G := GBasis(Ideal(x^7-x-1,x^3y-4x+1));
  G;



[
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36181199361y^4 - 5594389451313946416O/34524608236181199361y^3 - 5647365436133328098O/34524608236181199361y^2 - 27971979712025453040/34524608236181199361y - 40070468928802268986O/3452460823618
1199361,
  1/16384y^7 - 5/16384y^6 + 147/16384y^4 + 5/128y^3 - 31/16384y^2 + 17/128y - 20479/16384]
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```

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```
Use R ::= QQ[x,y], Lex;
  Set Indentation;  -- to improve the appearance of the output
  G := GBasis(Ideal(x^7-x-1,x^3y-4x+1));
  G;



[
  16028187571520907594440/34524608236181199361x - 4457540/5875764481y^7 - 47746460716124220/34524608236181199361y^6 + 8901757152713338440/34524608236181199361y^5 - 3541992534667352220/3452460
8236181199361y^4 - 55943894513139464160/34524608236181199361y^3 - 56473654361333280980/34524608236181199361y^2 - 27971979712025453040/34524608236181199361y - 400704689288022689860/345246082
36181199361,
  1/16384y^7 - 5/16384y^6 + 147/16384y^4 + 5/128y^3 - 31/16384y^2 + 17/128y - 20479/16384]
-------------------------------
 Len(G);

2
-------------------------------
  F := 16384*G[2];  -- clear denominators
  F;

y^7 - 5y^6 + 147y^4 + 640y^3 - 31y^2 + 2176y - 20479
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Document 1

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Next input

```
-- The current ring is R ::= QQ[x,y,z];
-------------------------------
$contrib/intprog.Man();

Suggested alias for this package:
    Alias IP := $contrib/intprog;

SYNTAX
    IP.TestSet(A: MAT,           Cost: LIST): TAGGED("IP")
    IP.TestSet(IP: TAGGED("IP"), Cost: LIST): TAGGED("IP")

    IP.Sol(A: MAT,           B: LIST, Cost: LIST);
    IP.Sol(IP: TAGGED("IP"), B: LIST, Cost: LIST);

DESCRIPTION
Let A be an NxM matrix with integer entries,
B a list with N integer entries,
C a list with M rational entries defining a linear cost function

        IP.TestSet(A, Cost)
    computes the toric ideal associated to A and a GBasis wrt a special
    order. This GBasis is a Test-Set for the Integer Programming
    problem IP c.
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-- The current ring is R ::= QQ[x,y,z];
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A := Mat([
      [1,0,0,0,0,0],
      [0,1,1,0,0,0],
      [0,0,0,1,0,0],
[0,0,0,0,1,1],
[1,1,0,0,1,0],[0,0,1,0,0,0],[0,0,0,1,0,1]]);
    B := [120,204,92,55,183,190,98];
    Cost := [1,2,1,2,3,1];
```

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Calculation in progress ...

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CONCLUSIONS

- The use of techniques from Groebner Basis Theory, and Commutative Algebra in general, to solve a classical problem from Economics, Integer Programming, shows how advances in research in pure mathematics find applications never imagined. As I am a double major, this project let me see first hand how a very abstract area of Mathematics has an application to my other field.