Musical Modulation by Symmetries

By Rob Burnham

Intro

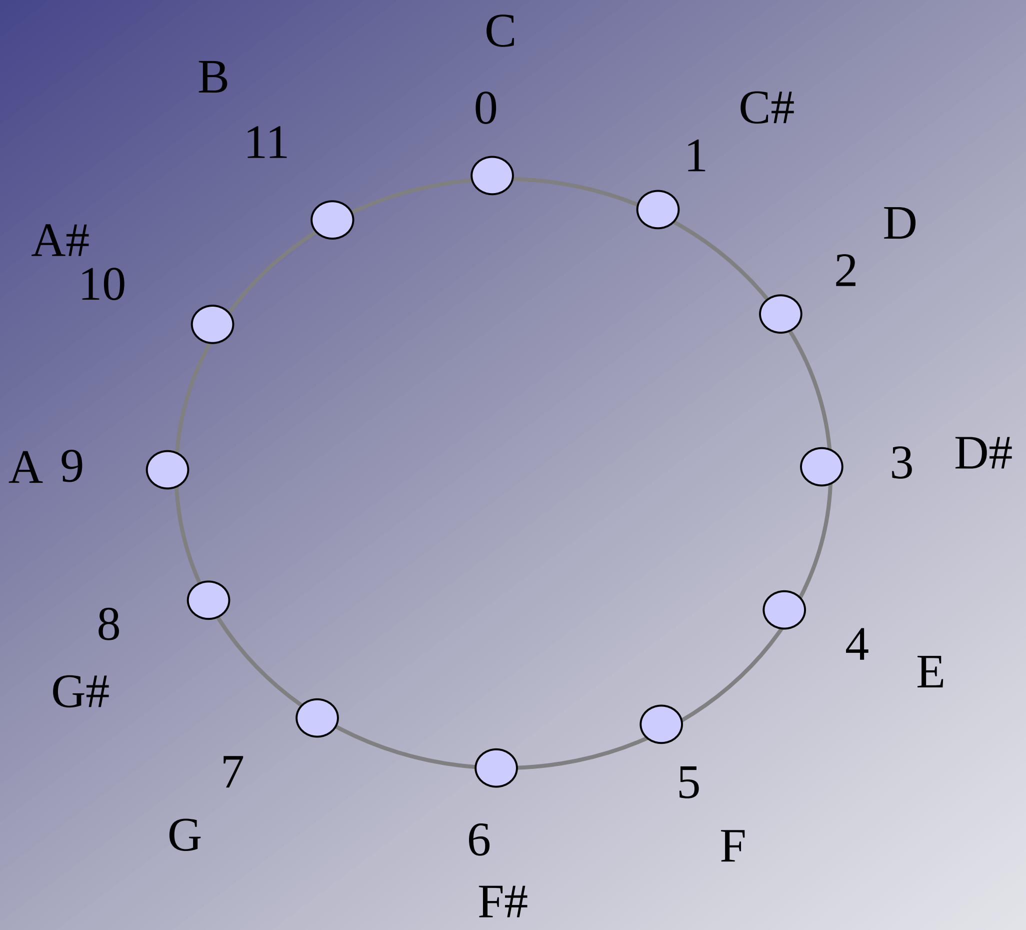
- Traditionally intervals have been conceived as ratios of frequencies. This comes from Pythagoras and his question about why "consonance" intervals like the octave, fifth are ratios of small integers.
  - Octave – 2/1
  - Fifth – 3/2

Intro Continued

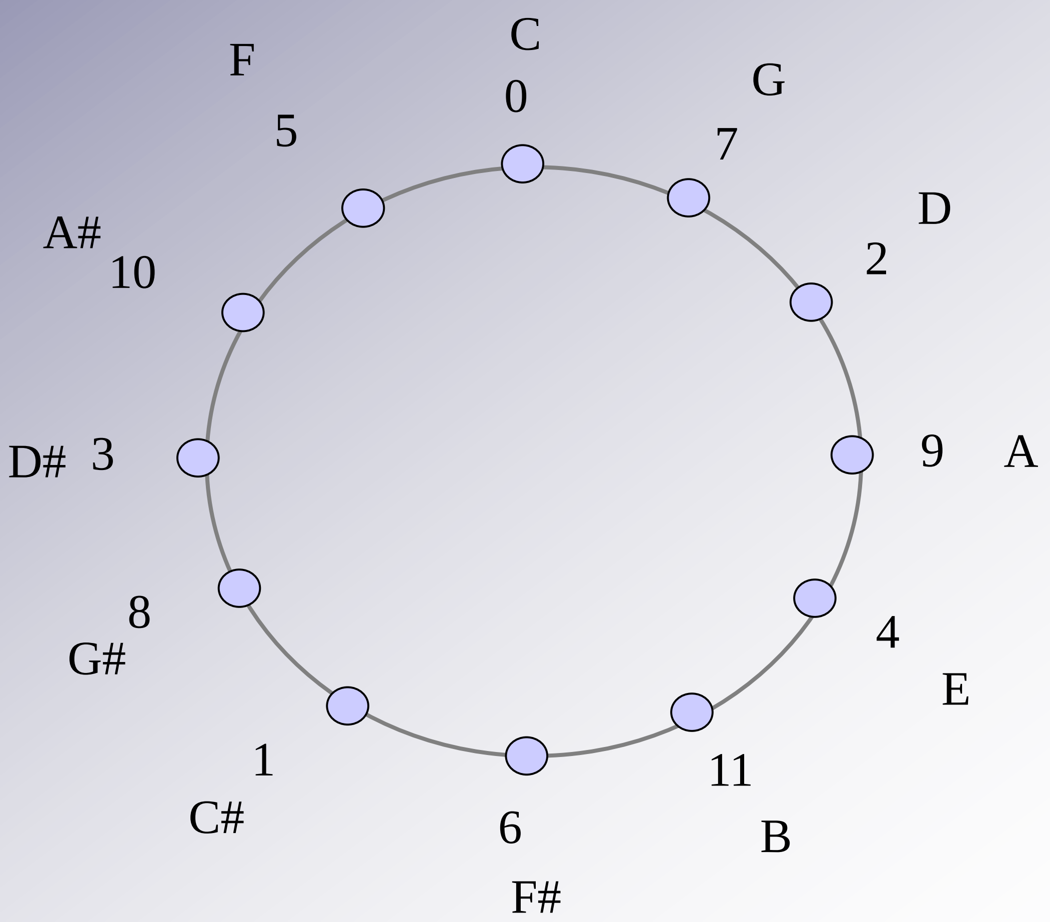
- My research bases itself on another way of modeling the 12 tone well tempered system(chromatic scale) that considers the individual intervals as transformations forming a mathematical group.

$Z_{12} \rightarrow C_{12}$ are Isomorphic

- Circle of Semi-tones

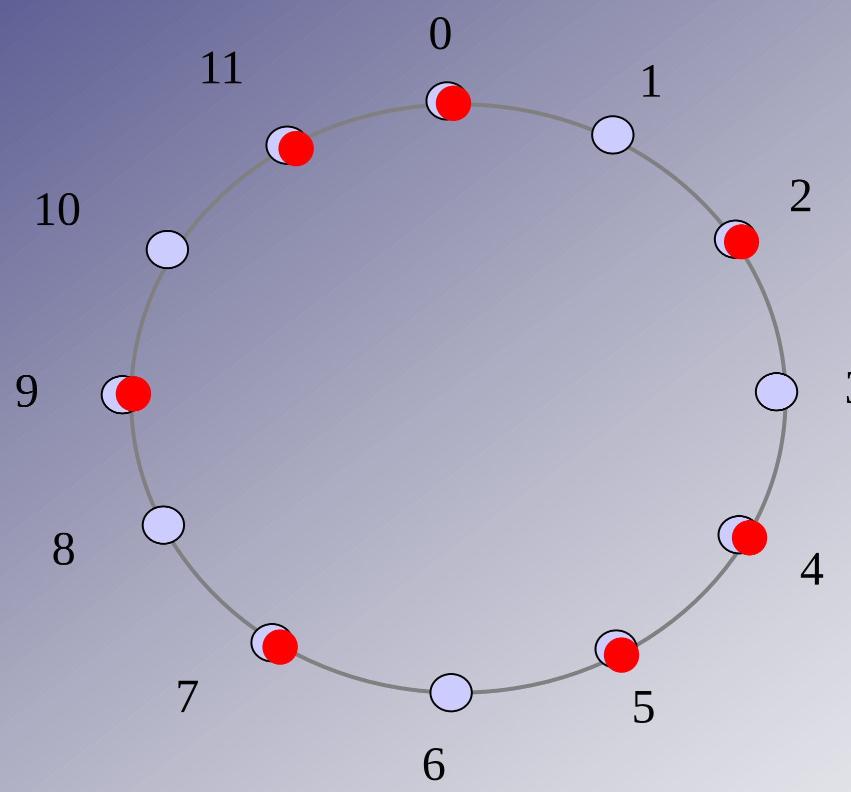


- Circle of Fifths

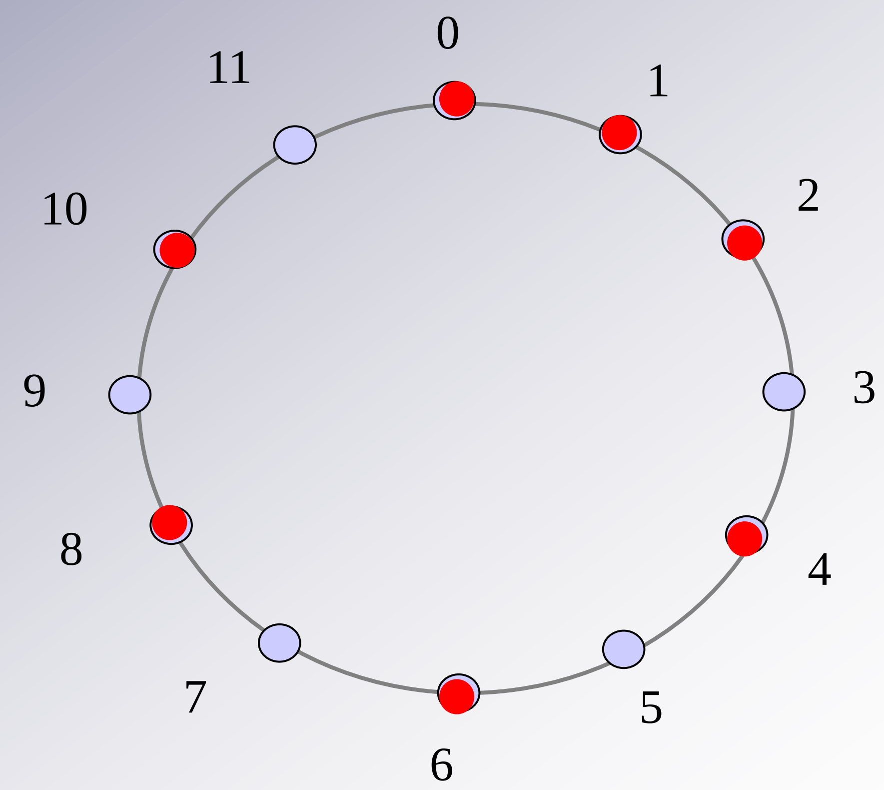


Seven note subset (NOT A SUBGROUP!!! 7 can not divide 12.) of C_{12}
The most well known are the diatonic major, melodic and harmonic minors.

- Diatonic C-Major Scale



- Scale # 62



My Research

- My research consists of taking an arbitrary 7 note subset of C_{12} (which for my presentation is scale #62) which we will call a “scale” generalizing the usual major and minor scales with their corresponding 12 tonalities.
- Using the model of musical modulation by symmetries developed by Mazzola and Muzzulini an analogy is made to modern physics, where symmetries are used to explain forces of transition.
- This modulation model provides direct modulations between all tonalities in C_{12} in 12 tempered tuning, and the relationships when applied to the diatonic major, harmonic and melodic minor scales agree with classical music theory. I extend this to an arbitrary scale consisting of seven tones.

Definitions

- For n=1 to 7, we define a triad of the n-th degree of a scale S (seven note subset of C_{12}) as $S_n = \{t_n, t_{n+2}, t_{n+4}\}$ where t_n is a tone of the seven note subset of C_{12}.

For Example

In the diatonic C-Major scale

– {0, 2, 4, 5, 7, 9, 11} = {C, D, E, F, G, A, B}

= $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$

$S_1 = \{t_1, t_3, t_5\}$ = {0, 4, 7} = {C, E, G}

$S_2 = \{t_2, t_4, t_6\}$ = {2, 5, 9} = {D, F, A}

$S_3 = \{t_3, t_5, t_7\}$ = {4, 7, 11} = {E, G, B}

and so on to S_7

Also

In scale #62

– {0, 1, 2, 4, 6, 8, 10} = {C, C#, D, E, F#, G#, A#}

= $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$

$S_1 = \{t_1, t_3, t_5\}$ = {0, 2, 6} = {C, D, F#}

$S_2 = \{t_2, t_4, t_6\}$ = {1, 4, 8} = {C#, E, G#}

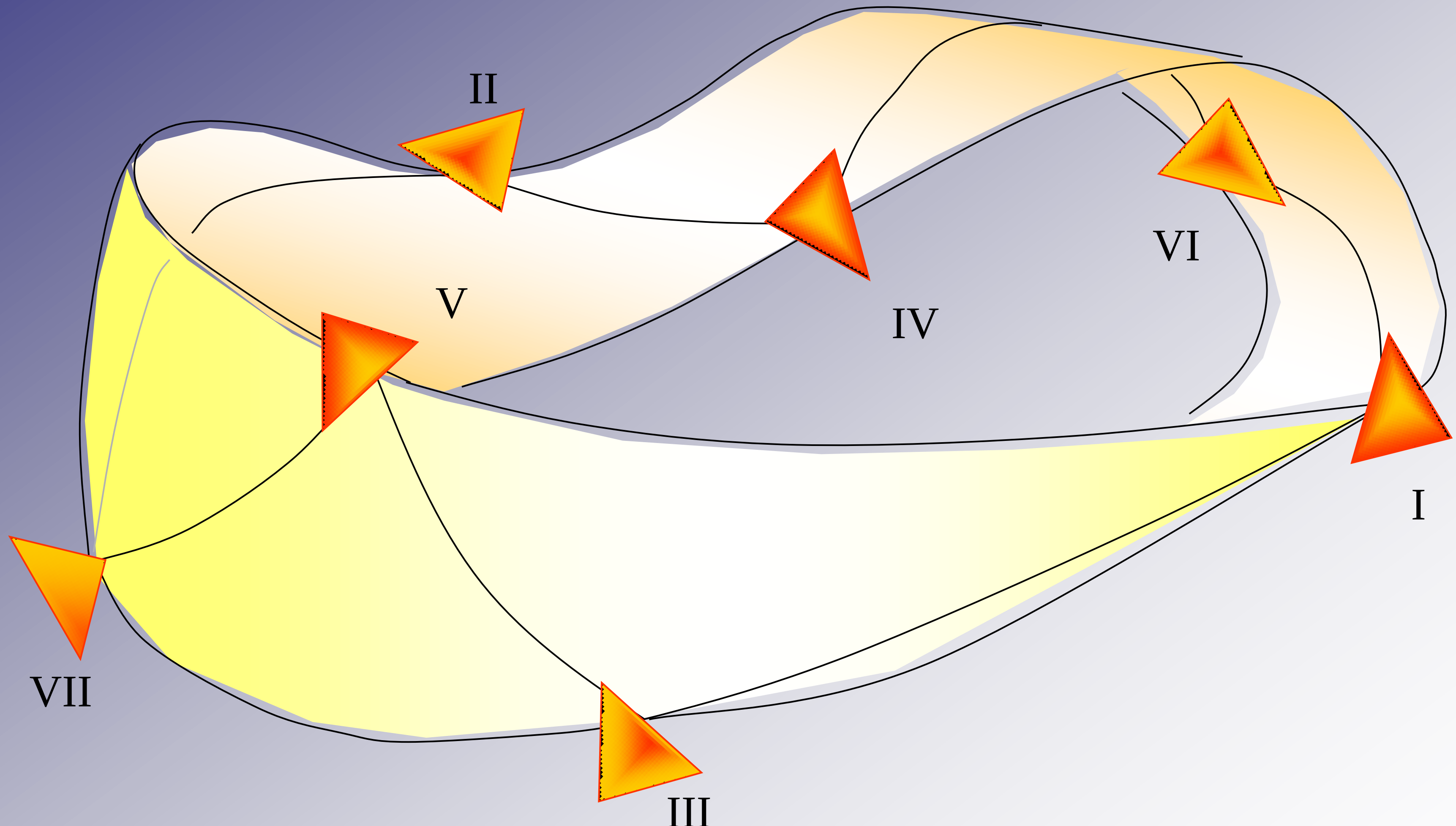
$S_3 = \{t_3, t_5, t_7\}$ = {2, 6, 10} = {D, F#, A#}

and so on to S_7

More Definitions

- A covering of a scale S by the seven triads S_1, S_2,..., S_7 is called the Triadic Interpretation of S and will be denoted $S^{(3)}$.
- The nerve of a triadic interpretation is a Möbius strip

Möbius Strip



Definition of Cadence Sets

- A subset X of triads of $S^{(3)}$(the triadic interpretation of S) is called a <u>cadence-set</u> of S if there is no other scale R in the same class such that the elements of X are also triads of $R^{(3)}$. The cadence-set X is called a <u>minimal cadence-set</u> if no proper subset of X is a cadence set.

Cadence sets

- Examples

  1.) A diatonic major scale has the following minimal cadence sets: {II, III}, {III, IV}, {IV, V}, {II, V}, {VII}.

  2.) For the harmonic minor scale, each pair of triads from its triadic interpretation forms a minimal cadence-set. Therefore, there are 21 different minimal cadence-sets for a harmonic minor scale.

38 Scale Orbits

- $C\binom{12}{7} = \frac{12!}{(7!)(5!)}$

There are 66 possible different 7-note subsets of C_{12} with different interval composition (different patterns). If we divide 792(C (12, 7)) by 12, we get 66 translation classes (we don't care about tonality, just scale pattern.) Under the action of the T/I group, which we will not explain at the moment. We can reduce our seven-note subsets to the representations of the 38 scale orbits.

Appendix 1

Scale orbits and number of quantized modulations

No.	*s*	(1)	(2)	(3)	(4)	
38	*******00000	$e^6 11$	9	42	54	(!)
38.1	*0*0**0*0*0*	$e^4 11$	5	20	26	
47	*0*****0*000	$e^8 11$	6	28	30	
47.1	*0**0*0*0*0*0	$e^2 11$	15	66	114	
50	**0***0**000	$e^8 11$	7	34	42	
50.1	***0***00*00	$e^6 11$	6	36	46	
52	***0*0***000	$e^8 11$	5	24	24	(!)
55	*****00*0*00	$e^4 11$	6	30	32	(!)
61	***00**0**00	$e^2 11$	10	38	62	
62	***0*0*0*0*0	$e^2 11$	5	24	24	(!)
39	******0*0000		9	29	93	
39.1	*0****0*0*00		6	23	55	
40	*****0**0000		10	24	108	
40.1	***0*0**0*00		7	26	72	
41	****0***0000		7	25	75	
41.1	****0*0**000		6	21	53	
42	******00*000		6	22	54	
42.1	***0**0*0*00		7	28	74	
43	*****0*0*000		6	22	57	
43.1	****0*0*0*00		7	26	72	
44	****0**0*000		9	23	89	
45	***0***0*000		7	21	63	
45.1	**0****0*000		10	21	105	
46	**0****0*000		6	26	56	
48	*****00**000		10	23	109	
48.1	***00***0*00		7	28	68	
49	***0**0**000		7	21	71	
49.1	**00****0*00		7	26	74	
51	****00***000		9	13	86	
53	*****0*00*00		7	27	67	
53.1	*0***0**0*00		9	25	91	
54	****0**00*00		7	32	71	
54.1	*0**0*0**00*		21	32	226	
56	**0***0*0*00		7	24	70	
57	****00**0*00		8	21	71	
58	**0**0**0*00		18	17	185	
59	**0*0***0*00		11	22	101	
60	***0**00**00		6	21	60	

(1) symmetry of $s^{(3)}$; (2) number of μ; (3) number of quanta; (4) number of quantized modulations; (!) not for every *p* quantized.

The numbering of the scale-orbits follows the numbering in [Mazzola 1990]. The scales of the orbits x and x.1 belong to the same orbit under the action of the *full* symmetry group of $\mathbf{Z}_{12}$.

- The entire line depicts the Chromatic Scale.
- * signifies the note is in the scale
- 0 signifies the note is not in the scale.

More definitions

In this context, Invertible Affine Transformations on C_{12} are called symmetries. They are written,

- T_nI_0 with n in C_{12} and I_0 is the inverse (for example, the inverse of 3 is 9).
- Example of a Translation,
  - $T_3\{0, 4, 7\} \rightarrow \{0+3, 4+3, 7+3\} \rightarrow \{3, 7, 10\}$
- Example of an Inverse
  - $I_0\{0, 4, 7\} \rightarrow \{0, 8, 5\}$

More Definitions

- The inner symmetries of a scale S are the symmetries that leave S invariant.
- For example T_4I_0 is a non-trivial inner symmetry of the C-major scale.

C-Major Scale – {0, 2, 4, 5, 7, 9, 11}

$\mathbf{T_4I_0\{0\} = T_4\{0\} = \{0+4\} = 4}$

$\mathbf{T_4I_0\{2\} = T_4\{10\} = \{10+4\} = 2}$

$\mathbf{T_4I_0\{4\} = T_4\{8\} = \{8+4\} = 12 = 0}$

$\mathbf{T_4I_0\{5\} = T_4\{7\} = \{7+4\} = 11}$

$\mathbf{T_4I_0\{7\} = T_4\{5\} = \{5+4\} = 9}$

$\mathbf{T_4I_0\{9\} = T_4\{3\} = \{3+4\} = 7}$

$\mathbf{T_4I_0\{11\} = T_4\{1\} = \{1+4\} = 5}$

As you can see using the inner symmetry we get the same scale back.

The Concept of Modulation in the Light of Symmetries.

Schoenberg defines modulation (referring to his book "Theory of Harmony")

- A
  - old key
  - neutral triads to weaken the old key
- B
  - new key
  - pivot root – progressions
  - to mark the turning point
- C
  - new key
  - Cadence to establish the new key

An Analogy

- In analogy to particle physics, we interpret modulations as hidden symmetries which are supported by a “quantum”. The explicit construction of the quantum will permit the calculation of the pivots.

Properties of the Modulator-quanta

- Modulators can be written in the form g =T_mf where f is an inner symmetry (T_nI_0) of $S^{(3)}$.

- Define Q to be particular subsets of Z_{12}, and let X be a minimal cadence-set of the target scale.
  - (1) There is a modulator g for $S^{(3)}$ and $R^{(3)}$ which is an inner symmetry of Q.
  - (2) All triads of X are subsets of Q.
  - (3) The only inner symmetry of $R \cap Q$ of the form T_nI_0, is the identity, and $R \cap Q$ is covered by triads of $R^{(3)}$.
  - (4) Q is a minimal set with properties (1) and (2).

Conclusions about the quantum and pivots found for the Diatonic Major Scale.

- When the quantum is found for the modulation from C-Major to F-Major, the pivot chords that are derived from Q coincide exactly with the chords specified by Schoenberg in his “Theory of Harmony” book. So the mathematical theory and procedure is coherent with classical music theory.

- Now for my research with Scale #62

The Scales and Triadic Interpretations of Scale #62

- C = {0, 1, 2, 4, 6, 8, 10} = {C, C#, D, E, F#, G#, A#}
- $S_{(C)}$ = {{0, 2, 6}, {1, 4, 8}, {2, 6, 10}, {4, 8, 0}, {6, 10, 1}, {8, 0, 2}, {10, 1, 4}}
- C# = {1, 2, 3, 5, 7, 9, 11} = {C#, D, D#, F, G, A, B}
- $S_{(C\#)}$ = {{1, 3, 7}, {2, 5, 9}, {3, 7, 11}, {5, 9, 1}, {7, 11, 2}, {9, 1, 3}, {11, 2, 5}}
- D = {2, 3, 4, 6, 8, 10, 0} = {D, D#, E, F#, G#, A#, C}
- $S_{(D)}$ = {{2, 4, 8}, {3, 6, 10}, {4, 8, 0}, {6, 10, 2}, {8, 0, 3}, {10, 2, 4}, {0, 3, 6}}
- D# = {3, 4, 5, 7, 9, 11, 1} = {D#, E, F, G, A, B, C#}
- $S_{(D\#)}$ = {{3,5,9}, {4, 7, 11}, {5, 9, 1}, {7, 11, 3}, {9, 1, 4}, {11, 3, 5}, {1, 4, 7}}
- E = {4, 5, 6, 8, 10, 0, 2} = {E, F, F#, G#, A#, C, D}
- $S_{(E)}$ = {{4, 6, 10}, {5, 8, 0}, {6, 10, 2}, {8, 0, 4}, {10, 2, 5}, {0, 4, 6}, {2, 5, 8}}
- F = {5, 6, 7, 9, 11, 1, 3} = {F, F#, G, A, B, C#, D#}
- $S_{(F)}$ = {{5, 7, 11}, {6, 9, 1}, {7, 11, 3}, {9, 1, 5}, {11, 3, 6}, {1, 5, 7}, {3, 6, 9}}

The Scales and Triadic Interpretations of Scale #62

- F# = {6, 7, 8, 10, 0, 2, 4} = {F#, G, G#, A#, C, D, E}
- $S_{(F\#)}$ = {{6, 8, 0}, {7, 10, 2}, {8, 0, 4}, {10, 2, 6}, {0, 4, 7}, {2, 6, 8}, {4, 7, 10}}
- G = {7, 8, 9, 11, 1, 3, 5} = {G, G#, A, B, C#, D#, F}
- $S_{(G)}$ = {{7, 9, 1}, {8, 11, 3}, {9, 1, 5}, {11, 3, 7}, {1, 5, 8}, {3, 7, 9}, {5, 8, 11}}
- G# = {8, 9, 10, 0, 2, 4, 6} = {G#, A, A#, C, D, E, F#}
- $S_{(G\#)}$ = {{8, 10, 2}, {9, 0, 4}, {10, 2, 6}, {0, 4, 8}, {2, 6, 9}, {4, 8, 10},{6, 9, 0}}
- A = {9, 10, 11, 1, 3, 5, 7} = {A, A#, B, C#, D#, F, G}
- $S_{(A)}$ = {{9, 11, 3}, {10, 1, 5}, {11, 3, 7}, {1, 5, 9}, {3, 7, 10}, {5, 9, 11}, {7, 10,1}}
- A# = {10, 11, 0, 2, 4, 6, 8} = {A#, B, C, D, E, F#, G#}
- $S_{(A\#)}$= {{10, 0, 4}, {11, 2, 6}, {0, 4, 8}, {2, 6, 10}, {4, 8, 11}, {6, 10, 0}, {8, 11,2}}
- B = {11, 0, 1, 3, 5, 7, 9} = {B, C, C#, D#, F, G, A}
- $S_{(B)}$ = {{11, 1, 5}, {0, 3, 7}, {1, 5, 9}, {3, 7, 11}, {5, 9, 0}, {7, 11, 1}, {9, 0, 3}}

The Minimal Cadence Sets of Scale #62

- {0, 2, 6} = {C, D, F#} = $I_{\text{"C"}}$
- {1, 4, 8} = {C#, E, G#} = $II_{\text{"C"}}$
- {6, 10, 1} = {F#, A#, C#} = $V_{\text{"C"}}$
- {8, 0, 2} = {G#, C, D} = $VI_{\text{"C"}}$
- {10, 1, 4} = {A#, C#, E} = $VII_{\text{"C"}}$

Translating the Minimal Cadence Sets from "C" to "F"

- For example,
- "C" → "F"
- $T_5(T_2I_0(C))$ = g (Our modulator)
  - $T_5(T_2I_0(\{0, 2, 6\})$ apply inverse

    $= T_5(T_2(\{0, 10, 6\})$ apply translation (n+2)

    $= T_5(\{2, 0, 8\})$ apply translation (n+5)

    $= \{1, 5, 7\} \rightarrow VI_{\text{"F"}}$

All the Cadence Sets for Scale #62 from "C" → "F"

- $I_{\text{"C"}}=\{0, 2, 6\} \rightarrow T_5(T_2I_0(\{0, 2, 6\}) = \{1, 5, 7\} = VI_{\text{"F"}}$
- $II_{\text{"C"}}=\{1, 4, 8\} \rightarrow T_5(T_2I_0(\{1, 4, 8\}) = \{11, 3, 6\} = V_{\text{"F"}}$
- $V_{\text{"C"}}= \{6, 10, 1\} \rightarrow T_5(T_2I_0(\{6, 10, 1\}) = \{6, 9, 1\} = II_{\text{"F"}}$
- $VI_{\text{"C"}}= \{8, 0, 2\} \rightarrow T_5(T_2I_0(\{8, 0, 2\}) = \{5, 7, 11\} = I_{\text{"F"}}$
- $VII_{\text{"C"}}= \{10, 1, 4\} \rightarrow T_5(T_2I_0(\{10, 1, 4\}) = \{3, 6, 9\} = VII_{\text{"F"}}$

Quantum

- Union the minimal cadence sets with our g applied to the same minimal cadence set.
- For example,
- $I_{"C"} = \{0, 2, 6\} \rightarrow \{1, 5, 7\}$ and then union this with $g(\{1, 5, 7\})$
  - $\{1, 5, 7\} \cup T_5(T_2 I_0(\{1, 5, 7\}))$ apply inverse
  - $\{1, 5, 7\} \cup T_5(T_2(\{11, 7, 5\}))$ apply translation (n+2)
  - $\{1, 5, 7\} \cup T_5(\{1, 9, 7\})$ apply translation (n+5)
  - $\{1, 5, 7\} \cup \{6, 2, 0\}$
  - $\{0, 1, 2, 5, 6, 7\}$

Candidates for Quantum of Scale #62

- 1.) {1, 5, 7} U $T_5(T_2I_0(\{1, 5, 7\})) \rightarrow \{0, 1, 2, 5, 6, 7\}$
- 2.) {11, 3, 6} U $T_5(T_2I_0(\{11, 3, 6\})) \rightarrow \{1, 3, 4, 6, 8, 11\}$
- 3.) {6, 9, 1} U $T_5(T_2I_0(\{6, 9, 1\})) \rightarrow \{1, 6, 9, 10\}$
- 4.) {5, 7, 11} U $T_5(T_2I_0(\{5, 7, 11\})) \rightarrow \{0, 2, 5, 7, 8, 11\}$
- 5.) {3, 6, 9} U $T_5(T_2I_0(\{3, 6, 9\})) \rightarrow \{1, 3, 4, 6, 9, 10\}$

All comply with property 1, which means that $T_5(T_2I_0)$ is a non-trivial inner symmetry of the candidates.

Intersections of Possible Quantum with the Target Scale

- Now intersect the quantum with the new scale (for us it is “F”) to find the intersection which contain the pivots.
  - For example,
    - $\{0, 1, 2, 5, 6, 7\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{1, 5, 6, 7\}$

All Intersections of Scale #62

- 1.) $\{0, 1, 2, 5, 6, 7\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{1, 5, 6, 7\}$
- 2.) $\{1, 3, 4, 6, 8, 11\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{1, 3, 6, 11\}$
- 3.) $\{1, 6, 9, 10\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{1, 6, 9\}$
- 4.) $\{0, 2, 5, 7, 8, 11\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{5, 7, 11\}$
- 5.) $\{1, 3, 4, 6, 9, 10\} \cap \{5, 6, 7, 9, 11, 1, 3\} = \{1, 3, 6, 9\}$
- This step is for checking property 3, which is that there is no non-trivial inner symmetry.

Conclusions about the Possible Quantum Scale #62

- By property 4 our best quantum is {1, 6, 9, 10}, because it is minimal and follows both properties 1 and 2.
- In this case there is only one pivot which is
  - {1, 6, 9} = {C#, F#, A} $\rightarrow$ $II_{\text{"F"}}$

Conclusion

- Since this general theory corresponds to Schoenberg's classical music theory with usual scales such as the diatonic major, this method when applied to an arbitrary scale, gives coherent results.

Proof

- There is a computer proof of the modulation theorem which guarantees the existence of at least one Quantum for every scale (7 note subset of C_{12}).