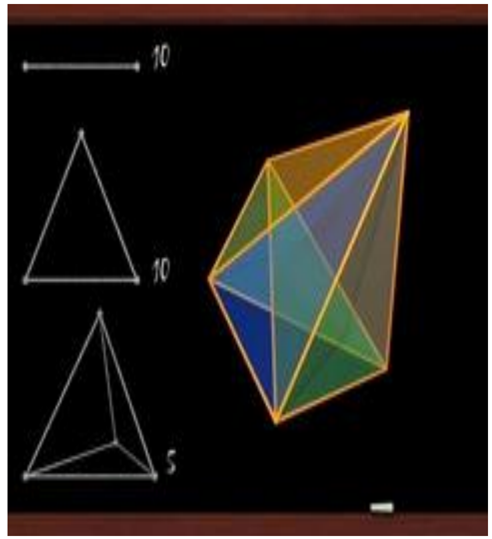
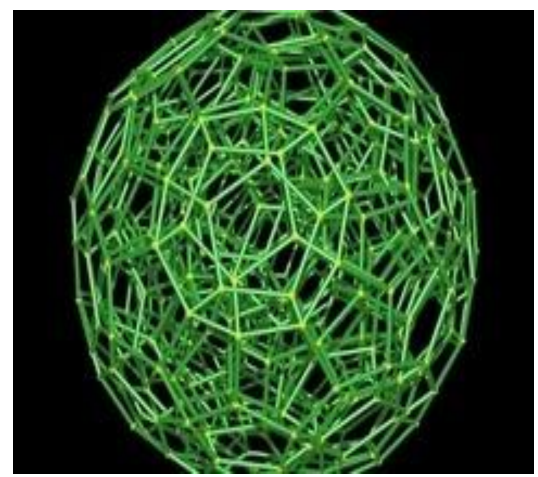
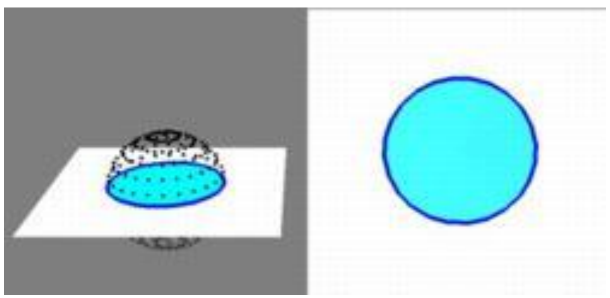
DIMENSION AND DIMENSIONAL ANALOGY

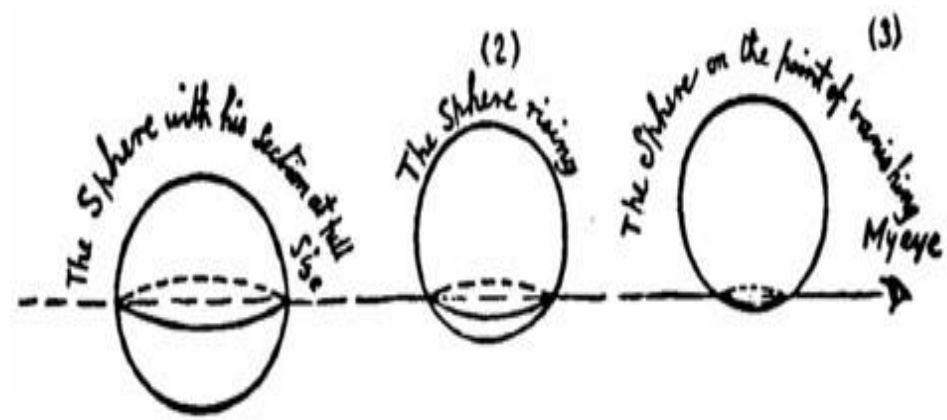




Over a hundred years ago, (1884), Edwin Abbott Abbott wrote a mathematical adventure set in a two-dimensional plane world, populated by a hierarchical society of **regular geometrical figures** (criminals are irregular, women are lines).

By imagining the **contact of beings from different dimensions**, the author fully exploited the power of **dimensional analogy.**





A Square in Flatland can understand a **sphere as a series of growing and shrinking circles**.

"O day and night, but this is wondrous strange"

FLATLAND

A ROMANCE OF MANY DIMENSIONS

No Dimensions

POINTLAND

One Dimension

LINELAND

Two Dimensions

FLATLAND

Three Dimensions

SPACELAND

"Fie, fie, how franticly I square my talk!"

As a first-rate fictional guide to the **concept of multiple dimensions of space**, the book also appeals to those who are interested in computer graphics.

It has been said that there is no better way to begin exploring **the problem of understanding higher-dimensional slicing phenomena** than reading this classic novel of the Victorian era.

As a satire, *Flatland* offered pointed observations on the social hierarchy of Victorian culture.

However, the novella's more enduring contribution is its **examination of dimensions**;

In a foreword to one of the many publications of the novella, noted science writer **Isaac Asimov described *Flatland* as "The best introduction one can find into the manner of perceiving dimensions."** As such, the novella is still **popular amongst mathematics, physics and computer science students.**

The book presents **dimensional analogy**, a fundamental **technique that lies at the basis of so many geometric ideas in mathematics**:

If we understand well what happens in **zero, one**, and **two** dimensions, then we are well on the way to understanding the **third**;

If we understand 0, 1, 2, and 3, then we can follow that momentum to the threshold of a **fourth** spatial dimension.

However, to understand the nature of four-dimensional space, dimensional analogy is *necessarily* employed.

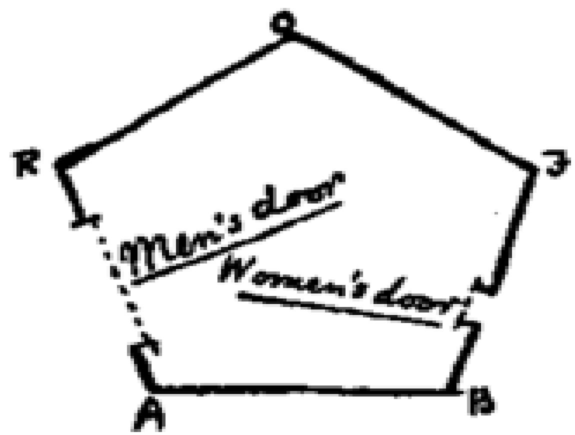
Dimensional analogy is the study of how **(n – 1) dimensions relate to n dimensions,** and then inferring how n **dimensions would relate to (n + 1) dimensions**.

If we can understand how a two-dimensional Flatlander conceives three-dimensional space, it is more feasible for us to conceive four-dimensional space.

We are taking the **fourth dimension as a spatial dimension, not as time**.

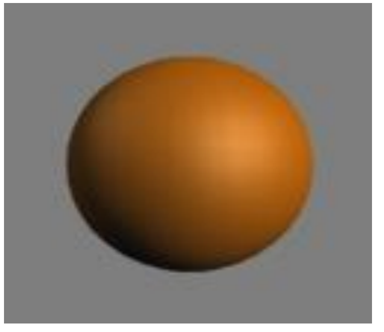
Flatland narrates a story about **A. Square**, who lives in a **two-dimensional world**, like the surface of a piece of paper (without any height).

From the perspective of this square, a three-dimensional being has seemingly god-like powers, such as being able to remove objects from a house without going through the door (by moving them "up and across" the third dimension).



This is because a **three-dimensional being moves in "another direction" (up-down), perpendicular to Flatland's plane,** which only has North-South and East-West

By applying dimensional analogy, one can infer that a four-dimensional being would be capable of similar feats from our three-dimensional perspective.



We could understand a **visit from a hyper-sphere from the fourth dimension in the same way a flatlander understands a visit from the third dimension**.

We would see a **small sphere appear and then grow**.

Once past the equator, the spheres begin to shrink again, until the sphere disappears as the hyper-sphere leaves our three-dimensional space again.

Rudy Rucker fictionalizes this in his novel **Spaceland,** in which the protagonist, **Joe Cube**, encounters four-dimensional beings who demonstrate such powers.

Momo, a creature from the fourth dimension, attaches a third eye to his brain, which lets him see the fourth dimension.

This novel is also a modern day social satire on life in Silicon Valley.

Why bother trying to visualize a higher-dimensional space that we can neither experience nor access directly? Besides pure curiosity, **4-D visualization has a wide variety of useful applications.**

Mathematicians have long wondered how to visualize 4-D space. In calculus, a very useful method of understanding functions is to *graph* them.

We can plot a real-valued function of one variable on a piece of graph paper, which is 2-D. We can also plot a real-valued function of two variables using a 3-D graph.

However, we run into trouble with even the simplest complex-valued function of 1 complex argument: **every complex number has two parts, the real part and the imaginary part, and requires 2 dimensions to be fully depicted.** This means that **we would need 4 dimensions to plot the graph of the complex function.** But, of course, there is no way to actually graph 4-D. However, it is believed that we can be trained to visualize 4-D by dimensional analogy.

Let's begin with the very basics. Let's start in a 1-D world.

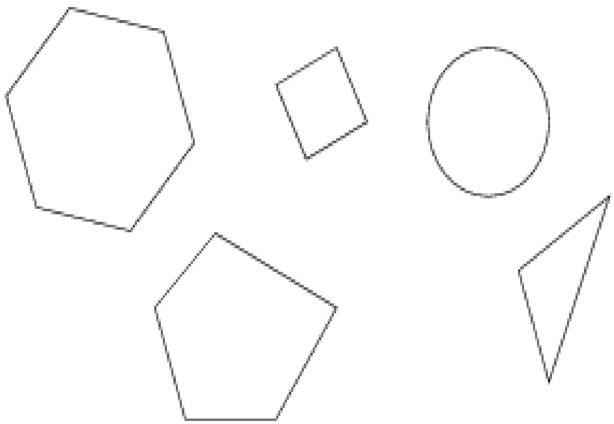
The 1-D world is like a piece of string.

The only dimension any object can have is length, because there aren't any other dimensions to accommodate width or height. So the only possible objects in 1-D are points, which are 0-D, and lines and curves, which are 1-D.

In order to completely specify a line segment, it is enough to specify its ***starting point*** and its ***ending point****.* In other words, the ***boundary*** of an object in 1-D consists of *points,* which are 0-D.

Now, let's move to the next higher dimension: 2-D. The 2-D world is a plane, like the surface of a piece of paper.

The 2-D world is much more interesting than the 1-D world, because a much larger variety of objects are possible. For example, we can have polygons and circles, in addition to points and lines:

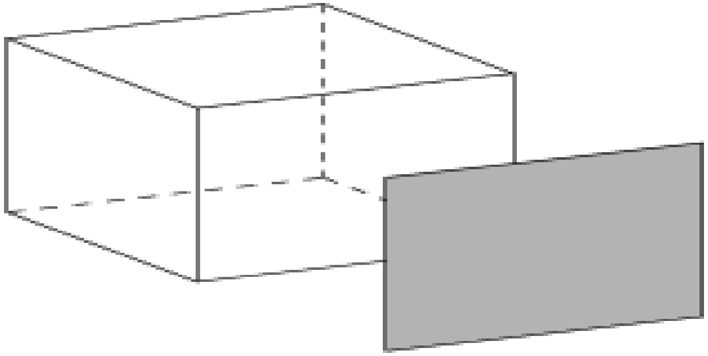


What is the **boundary** of a **polygon**? A polygon is bounded by **line segments, which are 1-D**.

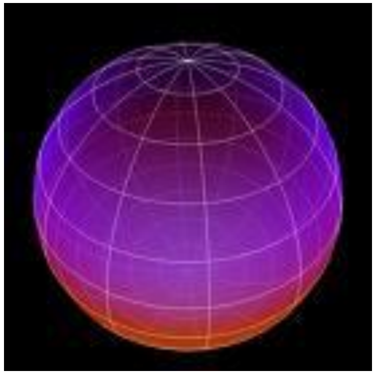
A circle also has a 1-D boundary, although it is a curved boundary. So **2-D objects are bounded by 1-D lines and curves**.

Now let's consider the 3-D world.

Objects in the 3-D world are bounded not by lines or curves, but by **2-D *surfaces***. For example, **a cube is bounded by 6 squares, and a ball is bounded by a spherical surface**.



The spherical surface is 2-D, because any point on the sphere is fully specified by only **two parameters**: longitude and latitude.



We can see a pattern emerge here.

Objects in 1-D are bounded by 0-D points, called **vertices.**

Objects in 2-D are bounded by 1-D lines or curves, called **edges**.

Objects in 3-D are bounded by 2-D surfaces called **faces.**

In other words, points in 1-D are *analogous* to lines and curves in 2-D: they form the **boundaries of objects in their respective dimensions.**

Similarly, lines and curves in 2-D are analogous to surfaces in 3-D. So, by applying dimensional analogy, we see that **in n dimensions, objects are bounded by (n-1)-dimensional boundaries.**

This leads us to conclude that **in 4-D, objects are bounded not by points, lines, or even surfaces, but *volumes.***

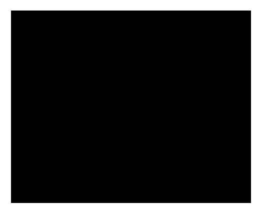
It would be rather difficult to realize this without applying dimensional analogy.

a 4-D cube is bounded by 8 cubes **(how do we know this?).** We call these bounding volumes the ***cells*** of the 4-D cube.

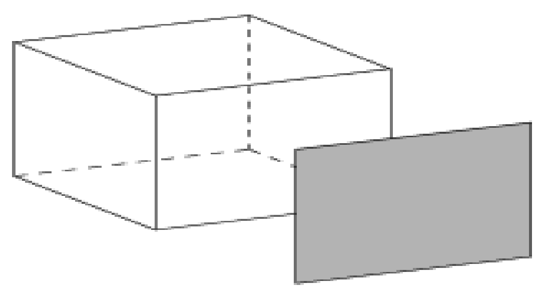
Arithmetic progression

A line has **2** points as its boundary. “**vertices**”

A square has **4** lines as its boundary. “**edges**”



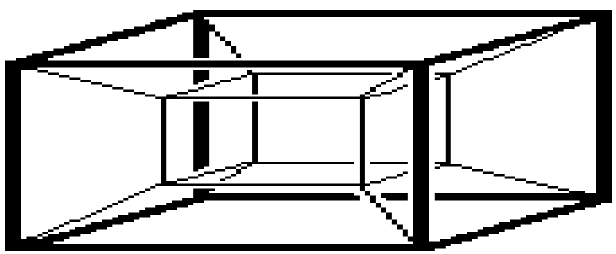
A cube has **6** squares as its boundary. “**faces**”

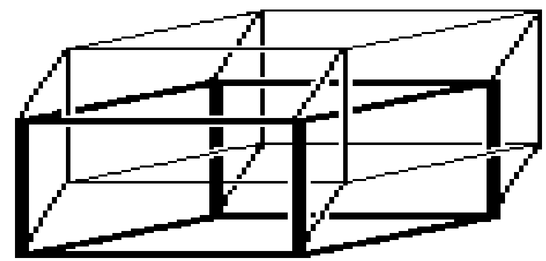


A hyper-cube has **8** cubes as its boundary. “**cells**”

HyperCube

A hypercube is a 4d object.
It is an example of a **polytope**





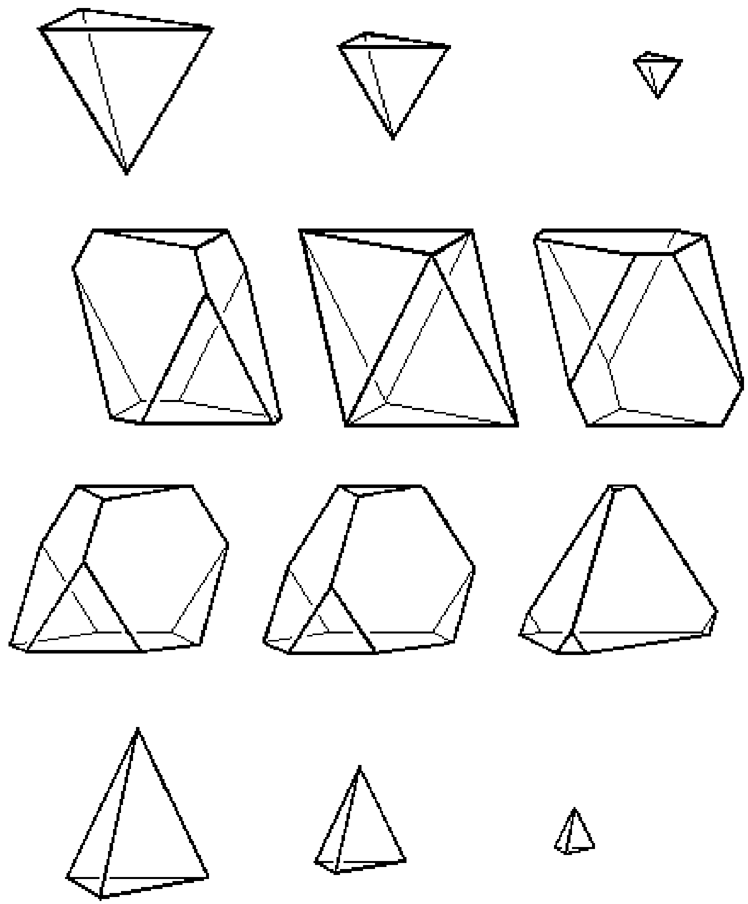
These are **3 dimensional looking 2 dimensional drawings of hyper-cubes**, called **tesseracts.**

Notice the one on the left.

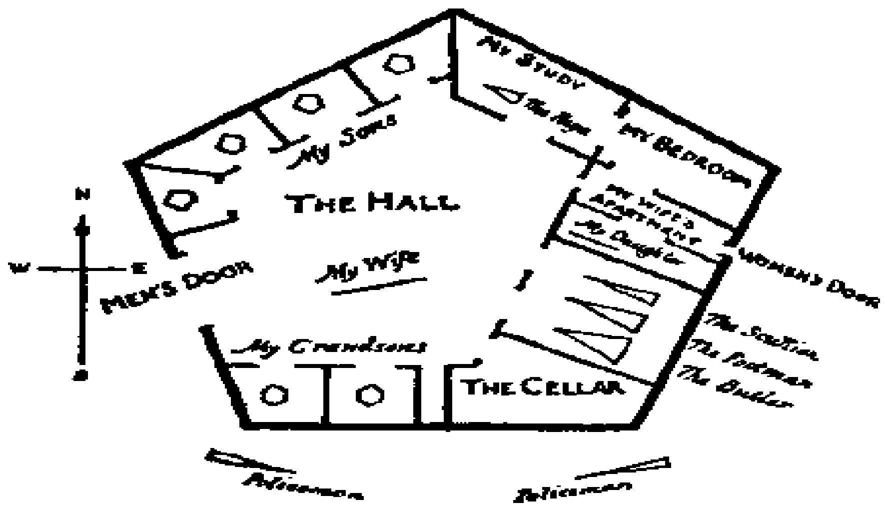
It is like 2 normal cubes(one smaller, one larger) which are connected at the corners.

Note that each cube is in different 3dimensional spaces and **they would be connected through the 4th dimension.**

3 D slices of the whole 4 D hyper-cube



If we look down upon a square in a flat, 2d plane, we can see the entire object a single glance. Only one perspective is needed. In fact, **we could place our finger inside the object without touching the sides.** This would be a profound feat for A. Square, a creature inhabiting Flatland. His house is one big pentagram and he can't just put his finger in the middle of the house without first "entering" through a door on one of the sides.

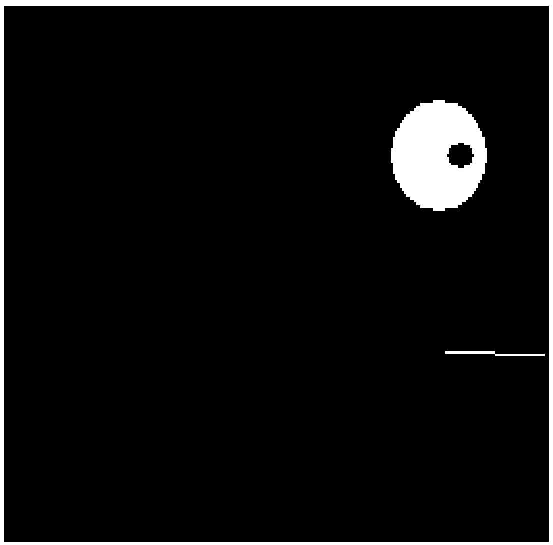


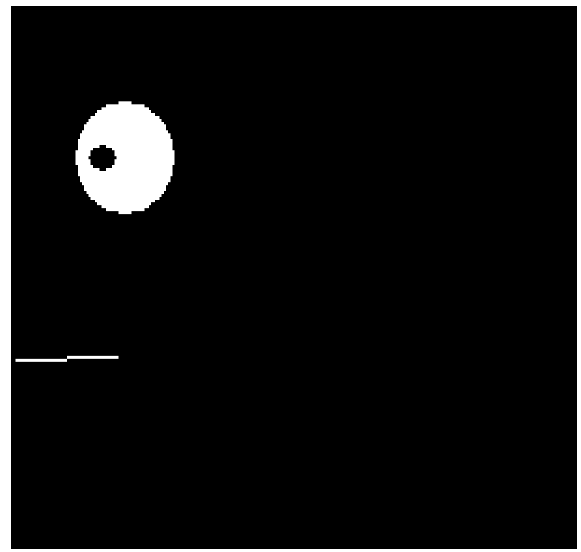
Analogously, fourth dimensional beings have the ability to visualize an entire cube at one glance. **Humans can only visualize one half the cube at any given second. Also, four dimensional beings could easily put their finger inside a closed cube without penetrating its sides**.

Other curiosities involve mirror images.

Imagine A. Square again. But now, lets pick him up off Flatland and put him back into the plane upside down.

He would now be the mirror image of his old self.





It is a bit tricky to imagine a human becoming a mirror image of themselves since we are unfamiliar with the fourth dimension rotation needed.

The “extrinsic” point of view

Curves and surfaces, in particular lines and planes, can be considered as **lying in a Euclidean space of higher dimension** (for example a plane in an ambient space of three dimensions).

The “intrinsic” point of view

the *intrinsic* point of view was developed, in which **one cannot speak of moving 'outside' the geometric object because it is considered as given in a self-contained way.**

The inhabitants of Flatland consider their plane from an intrinsic point of view. For them, there is no space, no third-dimension.

How do you consider your three-dimensional space, intrinsically or extrinsically?

SPHERELAND

The further mathematical fantasy, *Sphereland*, by Dionys Burger, published in 1962, revisits the world of Flatland in time to explore the mind-bending theories created by Albert Einstein, whose work so completely altered the scientific understanding of space, time, and matter.

Among Einstein's many challenges to common sense were the ideas of **curved space**, an **expanding universe** and the fact that light does not travel in a straight line.

Sphereland gives lay readers ways to start comprehending these confusing but fundamental questions of our reality.

In *Sphereland*, A. **Hexagon** (the grandson of A. Square) tells of his encounters with the **Sphere** and the Sphere's encounters with a fourth dimensional being.

It deals with such things as the **flipping of a object through a higher dimension** and the **curvature of the universe**.

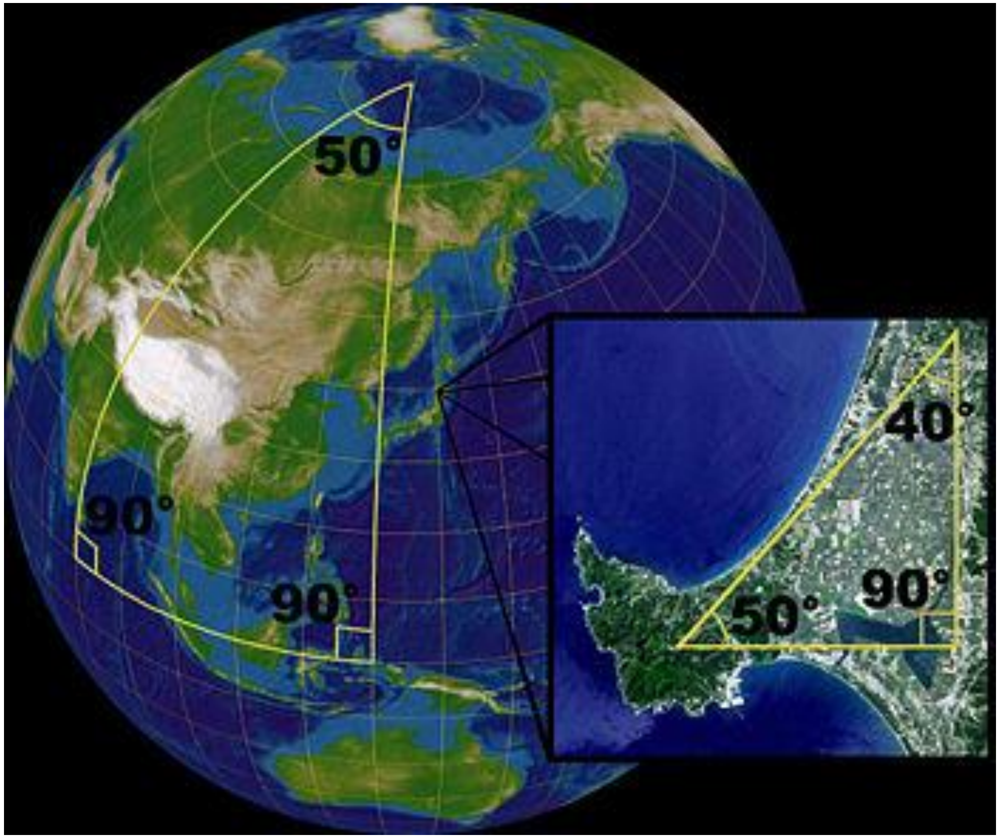
To any who are interested in understanding the nature of multiple spatial dimensions, by taking a step back to the **second dimension,** familiarizing ourselves with the world there and then looking at the **third dimension** that we are already familiar with, we can begin to have an understanding of what the **fourth spatial dimension** would mean to us.

When a prominent **surveyor** finds a **Triangle with more than 180** degrees, he is fired from his job and generally considered a crackpot, since such a construction is not possible in Euclidean Geometry.

He eventually makes friends with the grandson of A. Square, a **Hexagon**, because he is a mathematician and scientist. Together, they come upon a theory to explain the unusual measurements:

they actually live on a very **large sphere**, and the **Triangles have more than 180 degrees due to being inscribed on a non-planar surface.**

With help from the sphere from the first novel, they are able to prove this theory.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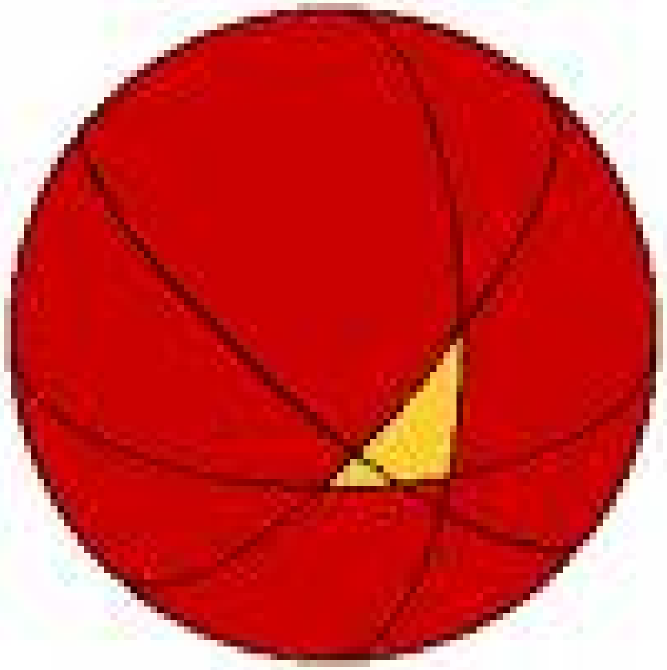


However, the established scientific community is not able to comprehend the idea proposed by the two, and thus they do not attempt to enlighten Flatland.

Furthermore, as the residents of Flatland advance, they begin to study space; they see distant worlds like their own, and the **surveyor tries to find the distance between their world and these distant worlds, using trigonometry and radar.**

From his calculations, he and the hexagon determine that the **universe is expanding;** again they try to reveal this theory to the outside world, but again it is not accepted.

Therefore, like his grandfather in the previous novel, the hexagon writes a book that is **not to be opened until the theory of the expanding universe is discovered and accepted by others**.

As with *Flatland* itself, the **Flatlanders' encounter with the third dimension** is intended to encourage consideration of **how higher dimensions would appear to us** and, in particular:

The concept that the universe is the three-dimensional surface of a four-dimensional hypersphere.

The Planiverse

The Planiverse is a book that features a group of college students and their teacher.

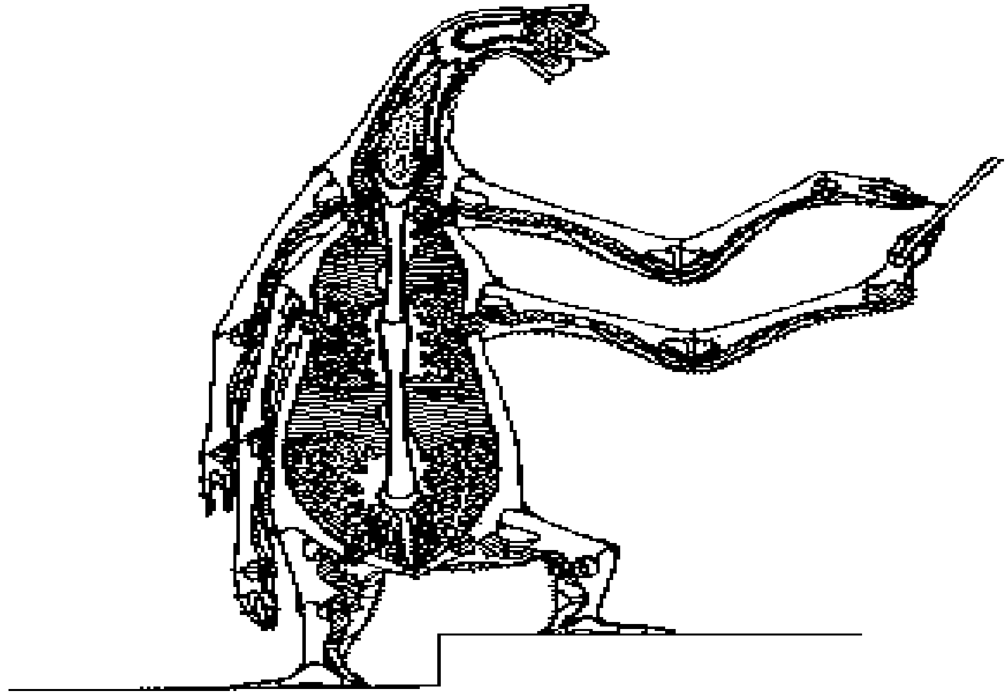
The students created an **artificial intelligence program that simulates creatures in their own world.. the "Planiverse"**.

They perform simple functions like “HUNT” and "F.E.C. IN FOCUS", which tells the user which *creature* they are looking at.

One day, one of the students focused on an F.E.C.. When *focused*, the F.E.C. replied with the word *YNDRD*.

This *YENDRED* seemed to be growing in intelligence.
As the group continues conversation with *YNDRD*, they come to realize that they have tapped in to an actual parallel universe.

The author creates a two-dimensional world near perfectly, with it's own physics, culture, and climate.



In *The Planiverse* we follow Yendred on the two-dimensional planet Arde.

Arde is different from Flatland, in that it is a vertical plane.

The **physics** of the Planiverse, the **geography** and **ecology** of Arde, and the **architecture**, **biology**, and **sociology** of the Ardeans are all worked out in great detail.

The book covers everything from **fishing boats** to **cities** to **space stations.**

Flatter Land

by Ian Stewart

This book focuses on **Non-Euclidean Geometry**.

Almost 100 years after A. Square's adventures in Spaceland that were related in *Flatland*, his great-great-granddaughter, Victoria Line, finds a copy of his book in her basement.

This prompts her to invite a visitor from Spaceland to visit her, but instead she is visited by the "Space Hopper"

The Space Hopper, more than being able to move between Flatland and Spaceland, can travel to any space in the *Mathiverse*, a set of all imaginable worlds.

After showing Vicky **higher dimensions**, he begins showing her more modern theories, such as **fractional dimensions** and **dimensions with isolated points**.

Topology and **hyperbolic geometry** are also discussed, as well as the **Projective "Plain"** (complete with **intersecting lions**) and the **quantum** level. Hopper and Victoria also visit the **Domain of the Hawk King** to discuss **time travel** and the **Theory of Relativity**

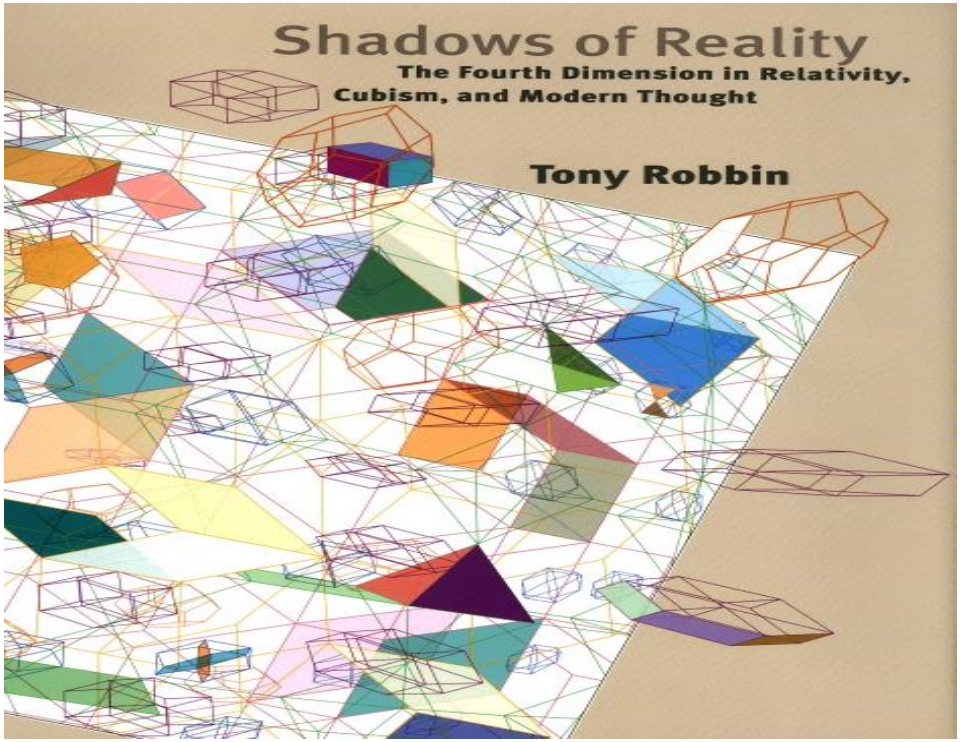
The Annotated Flatland: A Romance of Many Dimensions

Flatland distills what the Victorian era knew of higher mathematics--and then some--into a witty, complex novel of ideas.

Ian Stewart, in The Annotated Flatland, adds to Abbott's store of science the key discoveries made since.

The book does a superb job of explaining the original book's enigmas, allusions, ironies and implausibilities.

Among other things, Stewart comments on Abbott's comments on such things as the nature/nurture controversy, the fourth dimension and beyond, **the role of multidimensional spaces in economic systems, infinite series and perfect squares, celestial mechanics, and other matters close to the hearts of cosmologists and science buffs alike.**



Robbin explores the distinction between the **slicing**, or Flatland, model and the **projection**, or **shadow**, model

He reviews how **projective** ideas are the source of some of today's most exciting developments in **art, math, physics**, and **computer visualization**.

The Math Club's reading club

To be planned